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LETTER TO THE EDITOR

New exact integrable spin-1 quantum chains

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Abstract
New exactly integrable quantum spin-1 chains are derived. These new quantum Hamiltonians, like the XXZ chain, have a free parameter (anisotropy) and commute with the z-component of the total magnetization. The eigenvalues and eigenvectors are obtained directly by imposing a generalized coordinate Bethe ansatz. The derived Bethe ansatz equations have an unusual form and the associated R-matrix, has a dependence with the spectral parameter that is not of difference form, like the Hubbard quantum chain.

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FCA dedicates this work to the memory of Rinat Z Bariev, deceased 17 May 2001.

The anisotropic $S = \frac{1}{2}$ Heisenberg model, or *XXZ* chain, is one of the most studied quantum spin system in statistical mechanics. Since its exact solution, obtained by Yang and Yang [1], using directly the coordinate Bethe ansatz, this model has been considered as the classical example of the success of the Bethe ansatz. The advent of the quantum inverse scattering method [2,3], providing a powerful mathematical framework to the Bethe ansatz, enabled the discovery of several higher spin ($S > 1$) generalizations of the *XXZ* chain, preserving its exact integrability. The simplest of these generalizations are the spin-1 models having the z-component of the total magnetization as a good quantum number ($U(1)$ symmetry). The models discovered on this class are known in the literature as the Fateev–Zamolodchikov model [4], the Izergin–Korepin model [5] and the supersymmetric $OSP(1|2)$ model [6]. The integrability of these models is a consequence of the existence of an associated R-matrix satisfying the Yang–Baxter equation. In all the above models the associated R-matrix are regular, having a dependence on the spectral parameter of difference type. Recently an appropriate generalization of the coordinate Bethe ansatz [7,8] was discovered that allowed a direct solution for all the above models in an unified form. An earlier solution for the Izergin–Korepin model through the coordinate Bethe ansatz was obtained by Batchelor *et al* [9].

In this letter, directly using the coordinate Bethe ansatz we are going to show the existence of two new $U(1)$ symmetric spin-1 quantum chains. We start with a general spin-1 model with $U(1)$ symmetry and nearest neighbour interactions. Instead of writing this general Hamiltonian in terms of spin-1 Pauli matrices it is more convenient to write it in terms of the Weyl matrices $E^{l,m}$ $(l, m = 0, 1, 2)$, with i, j elements $E^{l,m})_{i,j} = \delta_{l,j}\delta_{m,j}$. We suppose that at each lattice site we may have zero particles $(S_z = -1)$, one particle $(S_z = 0)$ or two particles $(S_z = 1)$. The general Hamiltonian we consider, that conserves the number of particles, is given by

$$\begin{aligned} H = \sum_j \big[& E_j^{01}E_{j+1}^{10} + E_j^{10}E_{j+1}^{01} + t_1\big(E_j^{21}E_{j+1}^{01} + E_j^{12}E_{j+1}^{10}\big) + t_2\big(E_j^{01}E_{j+1}^{21} + E_j^{10}E_{j+1}^{12}\big) \\ & + t_p\big(E_j^{02}E_{j+1}^{20} + E_j^{20}E_{j+1}^{02}\big) + t_3\big(E_j^{12}E_{j+1}^{21} + E_j^{21}E_{j+1}^{12}\big) \\ & + v_{11}E_j^{11}E_{j+1}^{11} + uE_j^{22}E_{j+1}^{00} + v_{12}E_j^{11}E_{j+1}^{22} + v_{21}E_j^{22}E_{j+1}^{11} + v_{22}E_j^{22}E_{j+1}^{22}\big] \end{aligned} \tag{1}$$

where $t_1, t_2, t_3, t_{\rm p}$ are the non-diagonal hopping couplings, governing the spin motion and $u, v_{11}, v_{12}, v_{21}$ and v_{22} are the static potential energies. The couplings t_i $(i = 1, 2, 3)$ and $t_{\rm p}$ are the hopping terms for the single particle and pair of particles motion, respectively, while the two-, three- and four-particle static interactions are given by $u, v_{12}, v_{21}, v_{22}$. Certainly the Hamiltonian (1) is not integrable for an arbitrary choice of parameters. We want to find the constraints those parameters should satisfy in order the Hamiltonian (1) be solvable by the coordinate Bethe ansatz technique.

We consider the Hamiltonian (1) in a periodic lattice of L sites. This implies that besides the number of particles n $(0, 1, 2, \ldots)$, or z-magnetization in the spin language, the total momentum $p = \frac{2\pi}{L}l$ $(l = 0, 1, \ldots, L-1)$ is also a good quantum number. An arbitrary wavefunction $|\phi_n\rangle$, in the sector of n particles, is given by a linear combination of vectors $|x_1, x_2, \ldots, x_n\rangle$, corresponding to the configurations of n particles

$$|\phi_n\rangle = \sum_{1\leqslant x_1\leqslant x_2\leqslant\cdots\leqslant x_n\leqslant L}^{*} \Psi(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle \tag{2}$$

where the symbol (*) means that at most two coordinates may coincide at a given site. The ansatz that was applied with success in the known solvable cases of Hamiltonian (1) asserts that the amplitudes $\Psi(x_1, \ldots, x_2)$ in (2) should have the following form, if there exist no double occupation in any site:

$$\Psi(x_1, \ldots, x_n) = \sum_P A_{P_1\ldots P_n}^{1\ldots 1} \exp\left(\mathrm{i}\sum_{j=1}^n k_{P_j}x_j\right) \tag{3}$$

where $P = (P_1, \ldots, P_n)$ is the permutation of $1, 2, \ldots, n$ and k_j $(j = 1, \ldots, n)$ are unknown quasiparticle momenta. The n superscripts 1 in the amplitude indicate we have only single occupation of particles (no particles with spin $S_z = 1$). In the case where the lth and $(l+1)$th particles occupy the same position, the ansatz (3) is replaced by

$$\Psi(x_1, \ldots, x_l, x_l + 1, \ldots, x_n) = \sum_P A_{P_1\ldots P_l P_{l+1}\ldots P_n}^{1\ldots\overline{11}\ldots 1} \exp\left(\mathrm{i}\sum_{j=1}^n k_{P_j}x_j\right) \tag{4}$$

where the bar at the lth and $(l+1)$th positions of the superscript indicates the pair's position. The general case with many isolated particles and pairs follows from equations (3) and (4).

A direct substitution of the ansatz (3) for the amplitudes associated with the configurations where all the particles are isolated and not at nearest neighbour sites $(x_{i+1} > x_i + 1, i = 1, \ldots, L)$, fix the dependence of the eigenenergies on the unknown quasimomenta $\{k_i, i = 1, \ldots, n\}$, namely

$$E = 2\sum_{j=1}^n \cos k_j. \tag{5}$$

The equations for the amplitudes coming from the configurations where we have two particles located at consecutive sites, or the two particles are at the same site, forming a pair, are given by

$$S_{P_j P_{j+1}} A^{\dots 11\dots}_{\dots P_j P_{j+1}\dots} + S_{P_{j+1} P_j} A^{\dots 11\dots}_{\dots P_{j+1} P_j\dots} = 0 \tag{6}$$

and

$$N(y_j, y_{j+1}) A^{\dots 11\dots}_{\dots P_j P_{j+1}\dots} = C_1(y_j, y_{j+1}) A^{\dots \overline{11}\dots}_{\dots P_j P_{j+1}\dots} \tag{7}$$

respectively, where

$$S_{12} = C_0(y_1, y_2) D(y_1, y_2) + y_2[C_1(y_1, y_2) C_2(y_1, y_2) - v_{11} C_0(y_1, y_2)] \tag{8}$$

and

$$\begin{aligned} &C_0(y_1, y_2) = t_\mathrm{p} + u y_1 y_2 + t p y_1^2 y_2^2 - y_1 - y_2 - y_1^2 y_2 - y_1 y_2^2 \\ &C_1(y_1, y_2) = t_1 + t_2 y_1 y_2 \qquad C_2(y_1, y_2) = t_2 + t_1 y_1 y_2 \\ &D(y_1, y_2) = 1 + y_1 y_2 \qquad N(y_1, y_2) = D(y_1, y_2) - y_2 v_{11} \end{aligned} \tag{9}$$

and to simplify the notation we denote $y_j \equiv y_{P_j} \equiv \exp(\mathrm{i}k_{P_j})$. In contrast with the *XXZ* model the conditions (6) and (7) are not enough to prove the Bethe ansatz works. We have also to consider the eigenvalue equations in the case we have $n = 3$ and 4 particles at consecutive sites. For $n = 3$ particles, after substitution of (7), we obtain

$$\begin{aligned} &\sum_P \{d_1(y_1, y_2, y_3) N(y_1, y_2) C_1(y_1, y_2) + y_2 y_3 [t_1 C_1(y_2, y_3) - t_3 N(y_2, y_3)] \\ &\qquad \times C_1(y_1, y_2)\} C_1(y_1, y_3) A^{111}_{P_1 P_2 P_3} = 0 \\ &\sum_P \{d_2(y_1, y_2, y_3) C_1(y_1, y_2) N(y_2, y_3) + y_3 [t_2 C_1(y_1, y_2) - t_3 N(y_1, y_2)] \\ &\qquad \times C_1(y_2, y_3)\} C_1(y_1, y_3) A^{111}_{P_1 P_2 P_3} = 0 \end{aligned} \tag{10}$$

while for $n = 4$

$$\begin{aligned} &\sum_P \{[t_\mathrm{p}(1 + y_1 y_2 y_3 y_4) - (v_{22} - v_{11} - u) y_3 y_4] N(y_1, y_2) N(y_3, y_4) + t_1 y_2 y_3 y_4 \\ &\qquad \times C_1(y_1, y_2) N(y_3, y_4) + t_2 y_4 N(y_1, y_2) C_1(y_3, y_4)\} C_1(y_1, y_3) C_1(y_1, y_4) \\ &\qquad \times C_1(y_2, y_3) C_1(y_2, y_4) A^{1111}_{P_1 P_2 P_3 P_4} = 0 \end{aligned} \tag{11}$$

where

$$\begin{aligned} &d_1(y_1, y_2, y_3) = 1 + t_\mathrm{p} y_1 y_2 y_3 - (v_{21} - u/2) y_3 \\ &d_2(y_1, y_2, y_3) = t_\mathrm{p} + y_1 y_2 y_3 - (v_{12} - u/2) y_2 y_3. \end{aligned} \tag{12}$$

Following Baxter [10] the solution, if it exists, of equations (6)–(11), is given in terms of $S_{P_1 P_2}$. Apart of an overall multiplicative constant, that can be absorbed in the eigenvectors normalization, is given by

$$A^{11}_{P_1 P_2} = \epsilon_P S_{P_2 P_1} \qquad A^{111}_{P_1 P_2 P_3} = \epsilon_P S_{P_3 P_2} S_{P_3 P_1} S_{P_2 P_1} \tag{13}$$

$$A^{1111}_{P_1 P_2 P_3 P_4} = \epsilon_P S_{P_4 P_3} S_{P_4 P_2} S_{P_4 P_1} S_{P_3 P_2} S_{P_3 P_1} S_{P_2 P_1} \tag{14}$$

where $S_{P_i P_j}$ is given by (8) and $\epsilon_P = +1$ or -1 depending if the permutation is even or odd. Using relations (13) in (10) we obtain two complicated polynomial in the unknown variables y_1, y_2 and y_3, where the coefficients are functions of the coupling constants defining the general Hamiltonian (1). In order our Bethe ansatz works we should have all the coefficients of these polynomial identically zero, which will be true only for special values of the coupling constants in (1). Guessed by numerical calculations we were able to find besides the known solution of Fateev–Zamolodchikov [4], Izergin–Korepin [5], $OSP(1|2)$ model [6], two new solution ($\epsilon = +1, -1$), where the couplings appearing in (1) are given by

$$t_3 = \epsilon \qquad t_1 = \sqrt{t_{\mathrm{p}}^2 - 1}\exp\left(\mathrm{i}\frac{\pi}{3}\right) \qquad t_2 = -\epsilon\sqrt{t_{\mathrm{p}}^2 - 1}\exp\left(-\mathrm{i}\frac{\pi}{3}\right)$$
$$u = \epsilon t_{\mathrm{p}} + (2+\epsilon)t_{\mathrm{p}}^{-1} \qquad v_{11} = -\epsilon t_{\mathrm{p}} \qquad v_{12} = \frac{2-\epsilon}{2}t_{\mathrm{p}}^{-1} - \mathrm{i}\frac{\sqrt{3}}{2}\epsilon t_{\mathrm{p}} \tag{15}$$
$$v_{21} = \frac{2-\epsilon}{2}t_{\mathrm{p}}^{-1} + \mathrm{i}\frac{\sqrt{3}}{2}\epsilon t_{\mathrm{p}} \qquad v_{22} = (2-\epsilon)t_{\mathrm{p}}^{-1}$$

and t_{p} is a free complex parameter. In both cases ($\epsilon = \pm 1$) and equation (11) is satisfied by (14) as long as v_{22} takes the above value. The above results can now be generalized for arbitrary number $n > 4$ of particles, but since there is no new configuration, it is simple to convince ourselves that the Bethe ansatz works in general.

The Bethe ansatz equations, that fix the quasimomenta k_j, follows from the periodic boundary condition, in a standard procedure [10], and are given by

$$\exp^{\mathrm{i}k_j L} = (-)^{n-1}\prod_{l=1}^{n}\exp(-\mathrm{i}\theta(k_j, k_l)) \qquad j = 1, 2, \ldots, n \tag{16}$$

where

$$\exp(-\mathrm{i}\theta(k_j, k_l)) = \frac{S_{j,l}}{S_{l,j}} \tag{17}$$

and $S_{l,j}$ are given by (8). The solutions $\{k_j, j = 1, \ldots, n\}$ when inserted into (5) give the eigenenergies.

An important step towards the solution of (16) in the thermodynamic limit is achieved by introducing new suitable variables $\lambda_j = \lambda(k_j)$, that makes the phase shifts $\theta(k_j, k_l) = f(\lambda_j - \lambda_l)$ of difference form. Following [11] these new variables are found and the Bethe ansatz equations take the simple form

$$\mathrm{e}^{\mathrm{i}k_j L} = -\prod_{l=1}^{n}\frac{\sinh(\lambda_j - \lambda_l - \mathrm{i}\mu)}{\sinh(\lambda_j - \lambda_l + \mathrm{i}\mu)} \qquad j = 1, \ldots, n \tag{18}$$

where for the case $\epsilon = +1$,

$$\mathrm{e}^{\mathrm{i}k_j} = \frac{t_{\mathrm{p}}^{-1}\sinh\lambda_j \pm \mathrm{i}\sqrt{(4 - t_{\mathrm{p}}^{-2})\sinh^2\lambda_j + 3}}{2\sinh(\lambda_j + \mathrm{i}\frac{\pi}{3})} \qquad \mu = 2\pi/3 \tag{19}$$

and for the case $\epsilon = -1$

$$\mathrm{e}^{\mathrm{i}k_j} = \frac{t_{\mathrm{p}}^{-1}\sqrt{3}\sinh\lambda_j \pm \mathrm{i}\sqrt{(4 - 3t_{\mathrm{p}}^{-2})\sinh^2\lambda_j + 1}}{2\sinh(\lambda_j + \mathrm{i}\frac{\pi}{6})} \qquad \mu = \pi/3. \tag{20}$$

In both cases the energy and momentum are given by

$$E = \sum_{j=1}^{n} 2\cos(k_j) \qquad P = \sum_{j=1}^{n} k_j. \tag{21}$$

As we see form (1) and (15) the Hamiltonian of these new models, although having a real trace for real values of t_{p}, are not Hermitean. However, numerical results obtained by brute force diagonalization on small lattices show us that the low lying energies in the eigenspectra are all real, appearing as complex conjugated pairs of energies only in the upper part of the eigenspectra. This behaviour is quite similar to that appearing in the exact integrable spin-1 biquadratic model in a periodic lattice [12]. It is important to notice that the Bethe ansatz equations (18)–(20) are quite distinct from those of the known one-step exact integrable models. An extensive numerical analysis of these equations, for finite chains, is necessary in order to obtain the topology of its roots (particles, anti-particles, strings, etc). It is our plan to consider this problem and to present the phase diagram of these models in an extended version of this letter.

Finally, it is important to stress that although the other known exact integrable spin-1 models were derived by using the R-matrix approach, our new models merged from a direct application of the coordinate Bethe ansatz. A natural question that arises concerns the existence of an associated R-matrix for these models. According to the Reshetikihim criterion [3] any exact integrable Hamiltonian $H = \sum_i H_{i,i+1}$, with an associated R-matrix of difference form, should satisfy

$$[H_{i,i+1} + H_{i+1,i+2}, [H_{i,i+1}, H_{i+1,i+2}]] = W_{i,i+1} - W_{i+1,i+2} \tag{22}$$

where $W_{i,j}$ are arbitrary matrices associated with the sites i, j. We verified that our Hamiltonians (15) do not satisfy the above criterion and consequently they also do not have an associated R-matrix of difference form. This situation is similar to that of the Hubbard model [13] and the derivation of the associated R-matrix for the new models presented in this letter is an interesting open problem.

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